# Graph Based Quantum Error-Correcting Codes

Quantum error-correcting codes (QECC) are necessary to overcome decoherence in quantum computing and also essential in quantum communication. Calderbank, Rains, Shor, and Sloane in their seminal work [2] showed that finding binary quantum error-correcting codes (qubit codes) is equivalent to finding self-orthogonal additive codes over the finite field $\mathbb{F}_4 := \{0, 1, \omega, \overline{\omega} = \omega^2 = 1 + \omega\}$. A code $C$ is called additive if it is closed under addition but not necessarily under multiplication by the elements of $\mathbb{F}_4$. The *trace Hermitian inner product* of $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{F}_4^n$ is given by $\mathbf{x} * \mathbf{y} = \sum_{j=1}^{n} \left( x_j y_j^2 + x_j^2 y_j \right)$. Given an additive code $C$, its *symplectic dual code* $C^*$ is $C^* = \{\mathbf{x} \in \mathbb{F}_4^n : \mathbf{x} * \mathbf{c} = 0 \text{ for all } \mathbf{c} \in C\}$ and $C$ is said to be *(symplectic) self-dual* if $C = C^*$.

Schlingemann in [7] and subsequently by Danielsen [3] showed that every graph represents an additive code over $\mathbb{F}_4$ and every self-dual additive code over $\mathbb{F}_4$ can be represented by a graph. In particular, if $A(\Gamma)$ is the adjacency matrix of the graph $\Gamma$ and $I$ is the identity matrix then the additive $\mathbb{F}_4$ code $C(\Gamma)$ generated by the row span of the matrix $A(\Gamma) + \omega I$ is symplectic self-dual.

In the literature [4–6], majority of the best known zero-dimensional qubit codes were constructed from circulant graph based techniques. These types of codes were computationally straightforward to implement and provided better code parameters. Recently, Seneviratne and Ezerman [8, 9] used metacirculant graphs, a generalization of circulant graphs to construct record breaking qubit codes. We use the following combinatorial definition given in [1].

**Definition:** Let $m, n$ be two fixed positive integers and $\alpha \in \mathbb{Z}_n$ be a unit. Let $S_0, S_1, \ldots, S_{\lfloor m/2 \rfloor} \subseteq \mathbb{Z}_n$ satisfy the four properties; (a) $S_0 = -S_0$, (b) $0 \notin S_0$, (c) $\alpha^m S_k = S_k$ for $1 \leq k \leq \lfloor m/2 \rfloor$, and (d) If $m$ is even then $\alpha^{m/2} S_{m/2} = -S_{m/2}$. The meta-circulant graph $\Gamma := \Gamma\left(m, n, \alpha, S_0, S_1, \ldots, S_{\lfloor m/2 \rfloor}\right)$ has the vertex set $V(\Gamma) = \mathbb{Z}_m \times \mathbb{Z}_n$. Let $V_0, V_1, \ldots, V_{m-1}$, where $V_i := \{(i, j) : 0 \leq j \leq n - 1\}$ is a partition of $V(\Gamma)$. Let $1 \leq k \leq \lfloor m/2 \rfloor$. Vertices $(i, j)$ and $(i + k, h)$ are adjacent if and only if $(h - j) \in \alpha^i S_k$.

During the TADM-REU we will study open problems in this area and new code construction methods.

Projects in this area will require the use of Magma computational algebra system (CAS). The mentor will introduce the Magma CAS before the start of the TADM-REU, During the first week of the program an introduction to coding theory will be given. A background in linear algebra and discrete mathematics will be useful.

## References


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